

THE GRADIENT FLOW STRUCTURE FOR INCOMPRESSIBLE IMMISCIBLE TWO-PHASE FLOWS IN POROUS MEDIA

CLÉMENT CANCES, THOMAS O. GALLOUËT, AND LÉONARD MONSAINGEON

ABSTRACT. We show that the widely used model governing the motion of two incompressible immiscible fluids in a possibly heterogeneous porous medium has a formal gradient flow structure. More precisely, the fluid composition is governed by the gradient flow of some non-smooth energy. Starting from this energy together with a dissipation potential, we recover the celebrated Darcy-Muskat law and the capillary pressure law governing the flow thanks to the principle of least action. Our interpretation does not require the introduction of any algebraic transformation like, e.g., the global pressure or the Kirchhoff transform, and can be transposed to the case of more phases.

1. INTRODUCTION

1.1. General motivations. The models for multiphase porous media flows have been widely studied in the last decades since they are of great interest in several fields of applications, like e.g. oil-engineering, carbon dioxide sequestration, or nuclear waste repository management. We refer to the monographs [5, 6] for an extensive discussion on the derivation of models for porous media flows, and to [4, 11, 3, 13] for numerical and mathematical studies.

More recently, F. Otto showed in his seminal work [18] that the so-called *porous medium equation*:

$$\partial_t \rho - \Delta \rho^m = 0 \quad \text{for } (x, t) \in \mathbb{R}^N \times \mathbb{R}_+ \text{ and } m > 1,$$

which is a very simplified model corresponding to the case of an isentropic gas flowing within a porous medium, can be reinterpreted in a physically relevant way as the gradient flow of the free energy with respect to some Wasserstein metric in the space of Borel probability measures. Extensions to more general degenerate parabolic equations were then proposed for example in [1, 16]. See also for instance [7] or [15] for the interpretation of some dissipative systems as gradient flows in Wasserstein metrics.

In this note, we will focus on the model governing the motion of an incompressible immiscible two-phase flow in a possibly heterogeneous porous medium, that will appear in the sequel as (3) and (11)–(13). This model is relevant for instance for describing the flow of oil and water, whence the subscripts *o* and *w* appearing in the sequel of this note, within a rock that is possibly made of several rock-types. Our goal is to show that, at least formally, this model can be reinterpreted as the gradient flow of some singular energy. This will motivate the use of structure-preserving numerical methods inspired from [9] to this model in the future.

Our approach is inspired from the one of A. Mielke [17] and, more closely, to the one of M. A. Peletier [19]. The basic recipe for variational modeling is recalled in §1.2, then its ingredients are identified in §2. This approach is purely formal,

but it can be made rigorous under some unphysical strict positivity assumption on the phase mobilities η_o, η_w defined below. We will remain sloppy about regularity issues all along this note.

1.2. The recipe of getting formal variational models. Here we recall very briefly the main ingredients needed for defining a formal gradient flow.

- i. The *state space* \mathcal{M} is the set where the solution of the gradient flow can evolve.
- ii. At a point $\mathbf{s} \in \mathcal{M}$, the tangent space $T_{\mathbf{s}}\mathcal{M}$, to whom would belong $\partial_t \mathbf{s}$, is identified in a non-unique way with a so-called *process space* $\mathcal{Z}_{\mathbf{s}}$ (that might depend on \mathbf{s}). More precisely, we assume that for each $\mathbf{s} \in \mathcal{M}$ there exists an onto linear application $\mathcal{P}(\mathbf{s}) : \mathcal{Z}_{\mathbf{s}} \rightarrow T_{\mathbf{s}}\mathcal{M}$.
- iii. The *energy functional* $\mathcal{E} : \mathcal{M} \rightarrow \mathbb{R} \cup \{+\infty\}$ admits a (local) sub-differential $\partial_{\mathbf{s}}\mathcal{E}(\mathbf{s}) \subset (T_{\mathbf{s}}\mathcal{M})^*$ at $\mathbf{s} \in \mathcal{M}$.
- iv. The *dissipation potential* \mathcal{D} is such that, for all $\mathbf{s} \in \mathcal{M}$ and all $\mathbf{V} \in \mathcal{Z}_{\mathbf{s}}$, one has $\mathcal{D}(\mathbf{s}; \mathbf{V}) \geq 0$. It is supposed to be convex and coercive w.r.t. to its second variable.
- v. The initial data \mathbf{s}^0 belongs to \mathcal{M} .

All these ingredient being defined, we obtain from the *principle of least action* that $\mathbf{s} : \mathbb{R}_+ \rightarrow \mathcal{M}$ is the gradient flow of the energy \mathcal{E} for the dissipation \mathcal{D} if

$$(1a) \quad \partial_t \mathbf{s} = \mathcal{P}(\mathbf{s})\mathbf{V}$$

where

$$(1b) \quad \mathbf{V} \in \operatorname{argmin}_{\hat{\mathbf{V}} \in \mathcal{Z}_{\mathbf{s}}} \left(\max_{\mathbf{h} \in \partial_{\mathbf{s}}\mathcal{E}(\mathbf{s})} \left(\mathcal{D}(\mathbf{s}(t); \hat{\mathbf{V}}(t)) + \langle \mathbf{h}, \mathcal{P}(\mathbf{s})\hat{\mathbf{V}} \rangle_{(T_{\mathbf{s}}\mathcal{M})^*, T_{\mathbf{s}}\mathcal{M}} \right) \right).$$

The formula (1b) means that a gradient flow is lazy and smart: the motion aims to minimize the dissipation while maximizing the decay of the energy. We refer to [17, 19] for additional material on such a formal modeling and to [2] for an extensive (and rigorous) discussion on gradient flows in metric spaces.

2. VARIATIONAL MODELING FOR TWO-PHASE FLOWS IN POROUS MEDIA

2.1. State space and process space. Let Ω be an open subset of \mathbb{R}^N representing a (possibly heterogeneous) *porous medium*, let $\phi : \Omega \rightarrow (0, 1)$ be a measurable function (called *porosity*) such that $\underline{\phi} \leq \phi(\mathbf{x}) \leq \bar{\phi}$ for a.e. $\mathbf{x} \in \Omega$ for some constants $\underline{\phi}, \bar{\phi} \in (0, 1)$, and let $\underline{s}_o, \underline{s}_w : \Omega \rightarrow [0, 1]$ be two measurable functions (so-called *residual saturations*) such that $\underline{s}_o(\mathbf{x}) + \underline{s}_w(\mathbf{x}) < 1$ for a.e. $\mathbf{x} \in \Omega$. In what follows, we denote by

$$\bar{s}_o(\mathbf{x}) = 1 - \underline{s}_w(\mathbf{x}), \quad \bar{s}_w(\mathbf{x}) = 1 - \underline{s}_o(\mathbf{x}), \quad \text{for a.e. } \mathbf{x} \in \Omega.$$

For almost all $\mathbf{x} \in \Omega$, we denote by

$$\Delta_{\mathbf{x}} = \left\{ \mathbf{s} = (s_o, s_w) \in \mathbb{R}^2 \mid s_o + s_w = 1 \text{ with } \underline{s}_{\alpha}(\mathbf{x}) \leq s_{\alpha} \leq \bar{s}_{\alpha}(\mathbf{x}) \text{ for } \alpha \in \{o, w\} \right\}.$$

Let $\mathbf{s}^0 = (s_o^0, s_w^0)$ be a given initial saturation profile, we denote by m_{α} ($\alpha \in \{o, w\}$) the volume occupied by the phase α in the porous medium, i.e.,

$$m_o = \int_{\Omega} \phi(\mathbf{x}) s_o^0(\mathbf{x}) d\mathbf{x}, \quad \text{and} \quad m_w = \int_{\Omega} \phi(\mathbf{x}) s_w^0(\mathbf{x}) d\mathbf{x}.$$

For simplicity, we restrict our attention to the case where the volume of each phase is preserved: no source term and no-flux boundary conditions (otherwise, non-autonomous gradient flows should be considered). Hence the saturation profile lies at each time in the so-called state space \mathcal{M} , defined by

$$\mathcal{M} = \left\{ \mathbf{s} = (s_o, s_w) \mid s_\alpha : \Omega \rightarrow \mathbb{R}_+ \text{ with } \int_{\Omega} \phi(\mathbf{x}) s_\alpha(\mathbf{x}) d\mathbf{x} = m_\alpha \text{ for } \alpha \in \{o, w\} \right\}.$$

Let us now describe the processes that allow to transform the saturation profile. We denote by

$$\mathcal{Z}_{\mathbf{s}} = \left\{ \mathbf{V} = (\mathbf{v}_o, \mathbf{v}_w) \mid \mathbf{v}_\alpha : \Omega \rightarrow \mathbb{R}^N \text{ with } \mathbf{v}_\alpha \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\}$$

the *process space* of the admissible processes for modifying a saturation profile $\mathbf{s} \in \mathcal{M}$. The identification between $\mathbf{V} = (\mathbf{v}_o, \mathbf{v}_w) \in \mathcal{Z}_{\mathbf{s}}$ and $\dot{\mathbf{s}} = (\dot{s}_o, \dot{s}_w) \in T_{\mathbf{s}}\mathcal{M}$ is made through the onto operator $\mathcal{P}(\mathbf{s}) : \mathcal{Z}_{\mathbf{s}} \rightarrow T_{\mathbf{s}}\mathcal{M}$ defined by

$$(2) \quad \mathcal{P}(\mathbf{s})\mathbf{V} = \left(-\frac{1}{\phi} \nabla \cdot \mathbf{v}_o; -\frac{1}{\phi} \nabla \cdot \mathbf{v}_w \right), \quad \forall \mathbf{V} \in \mathcal{Z}_{\mathbf{s}}.$$

Since $\partial_t \mathbf{s} \in T_{\mathbf{s}}\mathcal{M}$, the relation (2) yields the existence of some *phase filtration speeds* $(\mathbf{v}_o, \mathbf{v}_w) \in \mathcal{Z}_{\mathbf{s}}$ such that the following *continuity equations* hold:

$$(3) \quad \phi \partial_t s_\alpha + \nabla \cdot \mathbf{v}_\alpha = 0, \quad \alpha \in \{o, w\}.$$

The relation (3) must be understood as the local volume conservation of each phase $\alpha \in \{o, w\}$. Finally, the duality bracket $\langle \cdot, \cdot \rangle_{(T_{\mathbf{s}}\mathcal{M})^*, T_{\mathbf{s}}\mathcal{M}}$ is given by

$$\begin{aligned} \langle \mathbf{h}, \dot{\mathbf{s}} \rangle_{(T_{\mathbf{s}}\mathcal{M})^*, T_{\mathbf{s}}\mathcal{M}} &= \sum_{\alpha \in \{o, w\}} \int_{\Omega} \phi h_\alpha \dot{s}_\alpha \\ &= - \sum_{\alpha \in \{o, w\}} \int_{\Omega} h_\alpha \nabla \cdot \mathbf{v}_\alpha = \sum_{\alpha \in \{o, w\}} \int_{\Omega} \nabla h_\alpha \cdot \mathbf{v}_\alpha. \end{aligned}$$

2.2. About the energy. For a.e. $\mathbf{x} \in \Omega$, we assume $\pi(\cdot, \mathbf{x}) : [\underline{s}_o(\mathbf{x}), \overline{s}_o(\mathbf{x})] \rightarrow \mathbb{R}$ to be a maximal monotone graph whose restriction $\pi|_{(\underline{s}_o, \overline{s}_o)}(\cdot, \mathbf{x})$ to the open interval $(\underline{s}_o(\mathbf{x}), \overline{s}_o(\mathbf{x}))$ is an increasing (single-valued) function belonging to $L^1(\underline{s}_o(\mathbf{x}), \overline{s}_o(\mathbf{x}))$. In particular, $\pi^{-1}(\cdot, \mathbf{x}) : \mathbb{R} \rightarrow [\underline{s}_o(\mathbf{x}), \overline{s}_o(\mathbf{x})]$ is a single valued function.

We denote by $\Pi : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ the (strictly convex w.r.t. its first variable) function defined by

$$\Pi(s_o, \mathbf{x}) = \begin{cases} \int_{\sigma(\mathbf{x})}^{s_o} \pi(a, \mathbf{x}) da - (\rho_o - \rho_w)sgz & \text{if } s_o \in [\underline{s}_o(\mathbf{x}), \overline{s}_o(\mathbf{x})], \\ +\infty & \text{otherwise,} \end{cases}$$

where, denoting by \mathbf{e}_z the downward unit normal vector of \mathbb{R}^N , we have set $z = \mathbf{x} \cdot \mathbf{e}_z$, and where g and ρ_α denote the gravity constant and the density of the phase α respectively, and where σ is such that $\mathbf{x} \mapsto \pi(\sigma(\mathbf{x}), \mathbf{x}) - (\rho_o - \rho_w)gz$ is constant. Since $\pi|_{(\underline{s}_o, \overline{s}_o)}(\cdot, \mathbf{x}) \in L^1(\underline{s}_o(\mathbf{x}), \overline{s}_o(\mathbf{x}))$, we get that $\Pi(\underline{s}_o(\mathbf{x}), \mathbf{x})$ and $\Pi(\overline{s}_o(\mathbf{x}), \mathbf{x})$ are finite for a.e. $\mathbf{x} \in \Omega$.

The *volume energy* function $E : \mathbb{R}^2 \times \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$(4) \quad E(\mathbf{s}, \mathbf{x}) = \begin{cases} \Pi(s_o, \mathbf{x}) & \text{if } \mathbf{s} = (s_o, s_w) \in \Delta_{\mathbf{x}}, \\ +\infty & \text{otherwise.} \end{cases}$$

The function $E(\cdot, \mathbf{x})$ is convex and finite on $\Delta_{\mathbf{x}}$ for a.e. $\mathbf{x} \in \Omega$. Its sub-differential is given by

$$\partial_{\mathbf{s}} E(\mathbf{s}, \mathbf{x}) = \begin{cases} \left\{ (h_o, h_w) \in \mathbb{R}^2 \mid h_o - h_w + (\rho_o - \rho_w)gz \in \pi(s_o, \mathbf{x}) \right\} & \text{if } \mathbf{s} \in \Delta_{\mathbf{x}}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Finally, we can define the so-called *global energy* $\mathcal{E} : \mathcal{M} \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$(5) \quad \mathcal{E}(\mathbf{s}) = \int_{\Omega} \phi(\mathbf{x}) E(\mathbf{s}(\mathbf{x}), \mathbf{x}) d\mathbf{x}, \quad \forall \mathbf{s} = (s_o, s_w) \in \mathcal{M}.$$

The saturation profile $\mathbf{s} \in \mathcal{M}$ is of finite energy $\mathcal{E}(\mathbf{s}) < \infty$ if and only if $\mathbf{s}(\mathbf{x}) \in \Delta_{\mathbf{x}}$ for a.e. $\mathbf{x} \in \Omega$. For $\mathbf{s} \in \mathcal{M}$ with finite energy one can check that the local sub-differential $\partial_{\mathbf{s}} \mathcal{E}(\mathbf{s})$ of \mathcal{E} at \mathbf{s} is given by

$$(6) \quad \partial_{\mathbf{s}} \mathcal{E}(\mathbf{s}) = \left\{ \mathbf{h} = (h_o, h_w) : \Omega \rightarrow \mathbb{R}^2 \text{ s.t.} \right. \\ \left. h_o - h_w + (\rho_o - \rho_w)gz \in \pi(s_o, \mathbf{x}) \text{ for a.e. } \mathbf{x} \in \Omega \right\}.$$

2.3. About the dissipation. The *permeability tensor* field $\mathbf{\Lambda} \in L^\infty(\Omega; \mathbb{R}^{N \times N})$ is assumed to be such that $\mathbf{\Lambda}(\mathbf{x})$ is a symmetric and positive matrix for a.e. $\mathbf{x} \in \Omega$. Moreover, we assume that there exist $\lambda_*, \lambda^* \in \mathbb{R}_+^*$ such that

$$\lambda_* |\mathbf{u}|^2 \leq \mathbf{\Lambda}(\mathbf{x}) \mathbf{u} \cdot \mathbf{u} \leq \lambda^* |\mathbf{u}|^2, \quad \text{for all } \mathbf{u} \in \mathbb{R}^N \text{ and a.e. } \mathbf{x} \in \Omega.$$

This ensures that $\mathbf{\Lambda}(\mathbf{x})$ is invertible for a.e. $\mathbf{x} \in \Omega$. Its inverse is denoted by $\mathbf{\Lambda}^{-1}(\mathbf{x})$.

We also need the two Carathéodory functions $\eta_o, \eta_w : \mathbb{R} \times \Omega \rightarrow \mathbb{R}_+$ — the so-called *phase mobilities* — such that $\eta_\alpha(\cdot, \mathbf{x})$ are Lipschitz continuous and non-decreasing on \mathbb{R}_+ for a.e. $\mathbf{x} \in \Omega$ and $\alpha \in \{o, w\}$. Moreover, we assume that $\eta_\alpha(s, \mathbf{x}) = 0$ if $s \leq \underline{s}_\alpha(\mathbf{x})$ and that $\eta_\alpha(s, \mathbf{x}) > 0$ if $s > \underline{s}_\alpha(\mathbf{x})$.

Given $\mathbf{s} = (s_o, s_w) \in \mathcal{M}$ and $\mathbf{V} = (\mathbf{v}_o, \mathbf{v}_w) \in \mathcal{Z}_{\mathbf{s}}$, we define the *dissipation potential* \mathcal{D} by

$$\mathcal{D}(\mathbf{s}, \mathbf{V}) = \frac{1}{2} \sum_{\alpha \in \{o, w\}} \int_{\Omega} \frac{\mathbf{\Lambda}^{-1} \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha}{\eta_\alpha(s_\alpha)} d\mathbf{x}, \quad \forall \mathbf{s} \in \mathcal{M}, \forall \mathbf{V} \in \mathcal{Z}_{\mathbf{s}}.$$

It is easy to check that dissipation is finite, i.e., $\mathcal{D}(\mathbf{s}, \mathbf{V}) < \infty$, iff $\mathbf{v}_\alpha = \mathbf{0}$ a.e. on $\{\mathbf{x} \in \Omega \mid s_\alpha(\mathbf{x}) \leq \underline{s}_\alpha(\mathbf{x})\}$.

2.4. Principle of least action and resulting equations. Let us consider the gradient flow governed by the energy \mathcal{E} , the continuity equation (3), and the dissipation \mathcal{D} . Let $\mathbf{s} \in \mathcal{M}$ be a finite energy saturation profile, then because of the *principle of least action* (1b) and of the definition (2) of the operator $\mathcal{P}(\mathbf{s}) : \mathcal{Z}_{\mathbf{s}} \rightarrow T_{\mathbf{s}}\mathcal{M}$, the process $\mathbf{V} = (\mathbf{v}_o, \mathbf{v}_w) \in \mathcal{Z}_{\mathbf{s}}$ and the *hydrostatic phase pressures* $\mathbf{h} = (h_o, h_w)$ must be chosen so that (\mathbf{V}, \mathbf{h}) is the min – max saddle-point of the functional

$$(7) \quad (\hat{\mathbf{V}}, \hat{\mathbf{h}}) \mapsto \mathcal{D}(\mathbf{s}, \hat{\mathbf{V}}) - \sum_{\alpha \in \{o, w\}} \int_{\Omega} \hat{h}_\alpha \nabla \cdot \hat{\mathbf{v}}_\alpha d\mathbf{x}.$$

One can first fix $\hat{\mathbf{h}} \in \partial_s \mathcal{E}(\mathbf{s})$ and minimize w.r.t. \mathbf{V} . This provides

$$(8) \quad \operatorname{argmin}_{\hat{\mathbf{V}} \in \mathcal{Z}} \left(\mathcal{D}(\mathbf{s}, \hat{\mathbf{V}}) - \sum_{\alpha \in \{o, w\}} \int_{\Omega} \hat{h}_{\alpha} \nabla \cdot \hat{\mathbf{v}}_{\alpha} d\mathbf{x} \right) = \left(-\eta_o(s_o) \mathbf{\Lambda} \nabla \hat{h}_o, -\eta_w(s_w) \mathbf{\Lambda} \nabla \hat{h}_w \right).$$

Injecting this expression in (7) and maximizing w.r.t. $\hat{\mathbf{h}} \in \partial_s \mathcal{E}(\mathbf{s})$, that is minimizing

$$(9) \quad \mathbf{h} = \operatorname{argmin}_{\hat{\mathbf{h}} \in \partial_s \mathcal{E}(\mathbf{s})} \left(\frac{1}{2} \int_{\Omega} \eta_{\alpha}(s_{\alpha}) \mathbf{\Lambda} \nabla \hat{h}_{\alpha} \cdot \nabla \hat{h}_{\alpha} \right)$$

among all elements $\hat{\mathbf{h}}$ in the subdifferential $\partial_s \mathcal{E}(\mathbf{s})$, yields

$$(10) \quad -\nabla \cdot (\mathbf{v}_o + \mathbf{v}_w) = 0, \quad \mathbf{v}_{\alpha} = -\eta_{\alpha}(s_{\alpha}) \mathbf{\Lambda} \nabla h_{\alpha}.$$

In (10) the first condition follows from the constraint $\hat{h} \in \partial_s \mathcal{E}(\mathbf{s})$ in (9), and the second one from (8).

Define the *phase pressures* $\mathbf{p} = (p_o, p_w)$ by $p_{\alpha}(\mathbf{x}) = h_{\alpha}(\mathbf{x}) + \rho_{\alpha} g z$, for a.e. $\mathbf{x} \in \Omega$ and $\alpha \in \{o, w\}$, then we recover the classical *Darcy-Muskat law*:

$$(11) \quad \mathbf{v}_{\alpha} = -\eta_{\alpha}(s_{\alpha}) \mathbf{\Lambda} \nabla (p_{\alpha} - \rho_{\alpha} g z), \quad \alpha \in \{o, w\}.$$

Moreover, it follows from (6) that the following *capillary pressure relation* holds:

$$(12) \quad p_o(\mathbf{x}) - p_w(\mathbf{x}) \in \pi(s_o(\mathbf{x}), \mathbf{x}) \quad \text{a.e. in } \Omega.$$

We recover here the multivalued capillary pressure relation proposed in [8, 10].

Combining (3) and (10) easily gives $\partial_t(s_o + s_w) = 0$, so that the condition

$$(13) \quad s_o + s_w = 1 \quad \text{a.e. in } \Omega,$$

is preserved along time and the whole pore volume remains saturated by the two fluids.

Gathering (3), (11), (12) and (13) gives the usual system of equations governing immiscible incompressible two-phase flows in porous media [5, 11, 3, 12, 10].

Remark 1. By similarity with the classical Wasserstein distance used in optimal mass transport [18] one could here endow the tangent space $T_s \mathcal{M}$ at $\mathbf{s} \in \mathcal{M}$ with a weighted \dot{H}^{-1} -scalar product

$$(\dot{\mathbf{s}}_1, \dot{\mathbf{s}}_2)_{T_s \mathcal{M}} = \sum_{\alpha \in \{o, w\}} \int_{\Omega} \eta_{\alpha}(s_{\alpha}) \mathbf{\Lambda} \nabla h_{1, \alpha} \cdot \nabla h_{2, \alpha} d\mathbf{x},$$

where, for $i \in \{1, 2\}$ and $\alpha \in \{o, w\}$, we have set $\dot{\mathbf{s}}_i = (\dot{s}_{i, o}, \dot{s}_{i, w})$ and where $h_{i, \alpha}$ solves

$$-\nabla \cdot (\eta_{\alpha}(s_{\alpha}) \mathbf{\Lambda} \nabla h_{i, \alpha}) = \dot{s}_{i, \alpha} \text{ in } \Omega, \quad \eta_{\alpha}(s_{\alpha}) \mathbf{\Lambda} \nabla h_{i, \alpha} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega.$$

Under some conditions on the functions η_{α} (see [14]), this should allow us to consider \mathcal{M} as a metric space endowed with the corresponding distance, but \mathcal{E} is not locally λ -convex for this Riemannian structure. The minimization (9) then consists in the selection of the subgradient with minimal norm.

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CLÉMENT CANCÈS (cances@ljl11.math.upmc.fr)

- (1) SORBONNE UNIVERSITÉS, UPMC UNIV PARIS 06, UMR 7598, LABORATOIRE JACQUES-LOUIS LIONS, F-75005, PARIS, FRANCE
- (2) CNRS, UMR 7598, LABORATOIRE JACQUES-LOUIS LIONS, F-75005, PARIS, FRANCE

THOMAS O. GALLOUËT (thomas.gallouet@inria.fr)

- (1) UNIVERSITÉ LIBRE DE BRUXELLES (ULB), BRUSSELS, BELGIUM
- (2) PROJECT-TEAM MEPHYSTO, INRIA LILLE - NORD EUROPE, VILLENEUVE D'ASCQ, FRANCE

LÉONARD MONSAINGEON (leonard.monsaingeon@ist.utl.pt)

- (1) CAMGSD, INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA, AV. ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL